ON CERTAIN INTEGRALS

BY

A. ZYGMUND

The present paper consists of three parts. The first two treat certain problems left open on the one hand by Littlewood and Paley, on the other by Marcinkiewicz. It turns out that these problems have close connection with each other. The remaining part is entirely independent of the first two and deals with absolute summability A of series. Its inclusion here may be explained by the fact that the integral it leads to is a limiting case of a certain family of integrals, another member of which plays an important role in §1.

§1. On the function $g^*(\theta)$ of Littlewood and Paley

1. In what follows, $P(\rho, t)$ will denote the Poisson kernel

$$P(\rho, t) = \frac{1}{2}\cdot\frac{1-\rho^2}{1-2\rho\cos t+\rho^2}. \tag{1.1}$$

By C we shall denote positive absolute constants, not necessarily always the same at every occurrence. Capital letters A, A', A^*, B, C, and so on, with subscripts, will mean positive coefficients depending only on the parameters explicitly stated (for example, $C_{\lambda,\sigma}$).

Let $f(\theta)$ be a function of period 2π and of the Lebesgue class L^r, $r \geqq 1$, and let

$$f(\theta) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta).$$

Let $f(\rho, \theta)$ be the Poisson integral of f,

$$f(\rho, \theta) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)\rho^\nu = \frac{1}{\pi}\int_0^{2\pi} f(\theta+t)P(\rho, t)dt.$$

By $\phi(z)$ we shall denote the analytic function, the real part of which is $f(\rho, \theta)$. To fix the ideas, it will be assumed that the imaginary part of $\phi(z)$ vanishes at the origin. Thus

$$f(\rho, \theta) = \Re\phi(z), \qquad \Im\phi(0) = 0 \qquad (z = \rho e^{i\theta}).$$

In their investigation on Fourier series Littlewood and Paley introduced two functions, $g(\theta)$ and $g^*(\theta)$[1]. The first of these is defined by the formula

Presented to the Society, October 30, 1943; received by the editors August 17, 1943.

(1) Littlewood and Paley [5]. Numbers in brackets refer to the References listed at the end of the paper.

$$g(\theta) = \left\{\int_0^1 (1-\rho)\,|\,\phi'(\rho e^{i\theta})\,|^2 d\rho\right\}^{1/2}, \qquad 0 \leqq \theta \leqq 2\pi. \tag{1.2}$$

As shown by the same authors, the function $g(\theta)$ satisfies for every $r>1$ the inequalities

$$A_r\left(\int_0^{2\pi} |f|^r d\theta\right)^{1/r} \leqq \left(\int_0^{2\pi} g^r d\theta\right)^{1/r} \leqq A_r\left(\int_0^{2\pi} |f|^r d\theta\right)^{1/r}, \tag{1.3}$$

of which the second indicates, in particular, that $g(\theta)$ is finite almost everywhere, a fact by no means obvious. It must be added that while the second inequality (1.3) is valid without any restriction on f, in the preceding inequality we assume that $\phi(0)=0$, that is, that the constant term $a_0/2$ of the Fourier series of f is equal to 0. Some such restriction is inevitable since the values of $g(\theta)$ do not depend on a_0, so that if a_0 is different from 0 and is sufficiently large, the first inequality in (1.3) is certainly false.

The definition of the function $g^*(\theta)$ of Littlewood and Paley is slightly less simple. It is given by the formula

$$g^*(\theta) = \left\{\int_0^1 (1-\rho)\chi^2(\rho, \theta) d\rho\right\}^{1/2}, \tag{1.4}$$

where

$$\chi(\rho, \theta) = \left\{\frac{1}{\pi}\int_0^{2\pi} |\,\phi'(\rho e^{i(\theta+t)})\,|^2 P(\rho, t) dt\right\}^{1/2}.$$

It may be expected that for $r>1$ the function $g^*(\theta)$ satisfies inequalities similar to (1.3), that is that

$$\left\{\int_0^{2\pi} |f|^r d\theta\right\}^{1/r} \leqq A_r^*\left\{\int_0^{2\pi} g^{*r} d\theta\right\}^{1/r}, \tag{1.5}$$

$$\left\{\int_0^{2\pi} g^{*r} d\theta\right\}^{1/r} \leqq A_r^*\left\{\int_0^{2\pi} |f|^r d\theta\right\}^{1/r}, \tag{1.6}$$

assuming in (1.5) that $a_0=0$.

The inequality (1.5) is an immediate consequence of the first inequality (1.3), since

$$g(\theta) \leqq 2g^*(\theta)\text{(2)}. \tag{1.7}$$

The inequality (1.6) was proved by Littlewood and Paley only for

$$r = 2, 4, 6, \cdots$$

and even for these values their proof is quite difficult. The purpose of this

(2) Loc. cit.

note is to prove the following theorem.

THEOREM 1. *The inequality* (1.6) *is valid for every* $r>1$.

The proof given below of this theorem is different from that of Littlewood and Paley. They used an argument modeled on the proof of the second inequality (1.3). The present proof reduces (1.5) to the second inequality (1.3).

2. It will be useful to precede the proof of the theorem by some remarks concerning the function $g^*(\theta)$.

That function played an important part in the proofs of Littlewood and Paley. Although it is possible to eliminate $g^*(\theta)$ from those proofs (and so to simplify the argument(3)), a study of that function seems to be of independent interest, for more than one reason.

For example, the function $g^*(\theta)$ intervenes in some estimates for the partial sums of Fourier series and it is quite likely that its presence there is essential(4).

Not less important is the fact that $g^*(\theta)$ is related to another function which has a very simple geometric significance.

For let Ω_0 denote any domain limited by a simple closed curve Γ_0, contained in the circle $|z|<1$, except for the point $z=1$ which Γ_0 has in common with the circumference

$$(\Gamma) \qquad |z| = 1.$$

We assume that Γ_0 is nontangential to Γ at the point $z=1$, that is to say that in the neighborhood of $z=1$ the curve Γ_0 is comprised between two chords of Γ through that point. Without loss of generality we may assume that Γ_0 is limited by the two tangents from $z=1$ to the circumference

$$|z| = 1 - \eta, \qquad 0 < \eta < 1,$$

and by the more distant arc of that circumference. For if η is small enough this domain will contain any other domain with the properties stated above.

It is easy to see that in this domain Ω_0 the expression

$$(2.1) \qquad \frac{1}{\rho}(1-\rho)P(\rho, t) = \frac{1}{\rho}\cdot\frac{1+\rho}{2}\cdot\frac{(1-\rho)^2}{(1-\rho)^2+4\rho\sin^2(t/2)}$$

exceeds a certain positive constant which depends only on η, and which we shall denote by $1/K_\eta^2$. Thus, if

$$\Omega_\theta = \Omega_{\eta,\theta}$$

denotes the domain Ω_0 rotated around $z=0$ by the angle θ, and if Ω_θ' is the part of the circle $|z|<1$ complementary to Ω_θ, then

(3) See A. Zygmund [10].

(4) Littlewood and Paley, loc. cit.

$$g^{*2}(\theta) = \left(\iint_{\Omega_\theta} + \iint_{\Omega'_\theta}\right) | \phi'(\rho e^{it}) |^2 (1 - \rho) P(\rho, t - \theta) d\rho dt \tag{2.2}$$
$$= g_1^{*2}(\theta) + g_2^{*2}(\theta),$$

say, and

$$\begin{aligned} g_1^{*2}(\theta) &= \iint_{\Omega_\theta} | \phi'(\rho e^{it}) |^2 \frac{1 - \rho}{\rho} P(\rho, \theta - t) \rho d\rho dt \\ &\geqq (1/K_\eta^2) \iint_{\Omega_\theta} | \phi'(\rho e^{it}) |^2 d\omega = (1/K_\eta^2) S^2(\theta), \end{aligned} \tag{2.3}$$

where $d\omega$ denotes the element of area, and

$$S(\theta) = S_\eta(\theta; \phi) = \left\{ \iint_{\Omega_\theta} | \phi'(\rho e^{it}) |^2 d\omega \right\}^{1/2} \tag{2.4}$$

Since $g^* \geqq g_1^*$, (2.3) implies

$$S(\theta) \leqq K_\eta g^*(\theta) \tag{2.5}$$

so that $g^*(\theta)$ is, essentially, a majorant of $S(\theta)$.

The geometric significance of the expression $S(\theta)$ is obvious: its square represents the area (multiple points counted according to their multiplicity) of the region obtained from Ω_θ by the transformation $w = \phi(z)$. The function $S(\theta)$ has an interesting property: any function $\phi(z)$ regular in $|z| < 1$ has a Stolz limit (that is to say, a limit along every nontangential path) at almost every point $z = e^{i\theta}$ of a set E of positive measure, if and only if $S(\theta)$ is finite almost everywhere in E(5).

The inequalities (1.6) and (2.5) imply

$$\left\{ \int_0^{2\pi} S^r(\theta) d\theta \right\}^{1/r} \leqq K_{r,\eta} \left\{ \int_0^{2\pi} | f(\theta) |^r d\theta \right\}^{1/r}, \qquad r > 1. \tag{2.6}$$

This inequality, however, is not new, and when written in the form

$$\left\{ \int_0^{2\pi} S^r(\theta) d\theta \right\}^{1/r} \leqq K_{r,\eta} \left\{ \int_0^{2\pi} | \phi(e^{i\theta}) |^r d\theta \right\}^{1/r} \text{(6)} \tag{2.7}$$

holds for any *positive* r(7), provided that $\phi(z)$ is of the class H^r, that is to say that

(5) For the necessity of this condition see Marcinkiewicz and Zygmund [7]. The sufficiency was recently proved by Spencer [9].

(6) The inequalities (2.6) and (2.7) are equivalent on account of the very well known theorem of M. Riesz concerning conjugate functions. See M. Riesz [8] or A. Zygmund [12. p. 147]. The latter book will in the sequel be quoted *TS*.

(7) See Marcinkiewicz and Zygmund [7].

$$\int_0^{2\pi} | \phi(\rho e^{i\theta}) |^r d\theta = O(1) \qquad \text{as } \rho \to 1.$$

Still another application of the function g^* will be given in §2.

3. Passing on to the proof of the theorem, let us take $\eta = 1/2$, so that from now on

$$\Omega_\theta = \Omega_{1/2,\theta}.$$

Thus the points of the region Ω_0' complementary to Ω_0 satisfy the inequalities

$$(3.1) \qquad 1/2 \leqq \rho < 1, \qquad |t| \geqq \alpha(1 - \rho),$$

where α is a positive constant whose numerical value is of no interest to us.

Let $\delta = 1 - \rho$. From the formula

$$P(\rho, t) = \frac{1}{2} \frac{(1 - \rho)(1 + \rho)}{(1 - \rho)^2 + 4\rho \sin^2 (t/2)}$$

we find that for $\rho \geqq 1/2$ the expression $P(\rho, t)$ does not exceed

$$\frac{\delta}{\delta^2 + 2t^2/\pi^2} \leqq \frac{\pi^2}{2} \frac{\delta}{\delta^2 + t^2} \leqq C \frac{\delta}{\delta^2 + t^2}.$$

This inequality holds true for $0 \leqq \rho < 1/2$, for there both $P(\rho, t)$ and $\delta/(\delta^2 + t^2)$ are limited above and below by positive numbers. Thus

$$(3.2) \qquad P(\rho, t) \leqq C\delta/(\delta^2 + t^2) \qquad (\delta = 1 - \rho, 0 \leqq \rho < 1).$$

From (2.2), (2.4), (3.2), and taking into account that η is now fixed, we get

$$(3.3) \qquad \begin{aligned} g^*(\theta) &\leqq CS(\theta) + C\left\{ \iint_{\Omega'_\theta} | \phi'(\rho e^{it}) |^2 \frac{\delta^2}{\delta^2 + (\theta - t)^2} d\rho dt \right\}^{1/2} \\ &= CS(\theta) + CG(\theta), \end{aligned}$$

say. Hence the inequality (1.6) will be a consequence of the inequalities

$$(3.4) \qquad \left\{ \int_0^{2\pi} S^r(\theta) d\theta \right\}^{1/r} \leqq A_r \left\{ \int_0^{2\pi} |\phi(e^{i\theta})|^r d\theta \right\}^{1/r},$$

$$(3.5) \qquad \left\{ \int_0^{2\pi} G^r(\theta) d\theta \right\}^{1/r} \leqq A_r \left\{ \int_0^{2\pi} |\phi(e^{i\theta})|^r d\theta \right\}^{1/r}.$$

Proving (3.5) will be our main task. Although, as was stated above, the inequality (3.4) is not new (cf. (2.7)), and its proof may be found in another paper, the knowledge of that paper is not indispensable here. The proofs of (3.4) and (3.5) run in parallel, and the argument which gives (3.5) completed by remarks will also give (3.4).

It will be necessary to consider a generalization of the integral $G(\theta)$ (cf. (3.3)). Plainly[8],

$$G(\theta) = \left\{ \iint_{\Omega'_0} |\phi'(\rho e^{i(\theta+t)})|^2 \frac{\delta^2 dt d\rho}{\delta^2 + t^2} \right\}^{1/2} \tag{3.6}$$
$$\leqq C \left\{ \int_{1/2}^{1} d\rho \int_{|t| \geqq \alpha\delta} |\phi'(\rho e^{i(\theta+t)})|^2 \frac{\delta^2}{t^2} \right\}^{1/2}$$

It is also easy to see that the last expression may not exceed $C_\alpha g^*(\theta)$.

Let σ be any number greater than 1. We introduce the function

$$G_\sigma(\theta) = G_\sigma(\theta; \phi) = \int_0^1 d\rho \delta^\sigma \int_{|t| \geqq \alpha\delta} \frac{|\phi'(\rho e^{i(\theta+t)})|^2}{|t|^\sigma} dt.$$

Thus

$$G(\theta) \leqq C G_2(\theta). \tag{3.7}$$

4. We require a few lemmas.

LEMMA 1. *Let $f(\theta)$ be of period 2π and of the class L^r, $r > 1$. The function*

$$f^*(\theta) = \max_{0<|h|\leqq\pi} \left| \frac{1}{h} \int_0^h |f(\theta + u)| du \right| \tag{4.1}$$

satisfies the inequality

$$\left\{ \int_0^{2\pi} f^{*r}(\theta) d\theta \right\}^{1/r} \leqq B_r \left\{ \int_0^{2\pi} |f(\theta)|^r d\theta \right\}^{1/r}$$

with $B_r \leqq 2r/(r-1)$.

This is a well known result of Hardy and Littlewood[9].

LEMMA 2. *Let $f(\rho, \theta)$ be the Poisson integral of the function $f \in L$. Then, with the notation* (4.1),

$$|f(\rho, \theta + u)| \leqq C f^*(\theta)(1 + |u|/\delta), \qquad \delta = 1 - \rho. \tag{4.2}$$

This lemma is also known[10]. For the applications we shall need it in a slightly more general form, which is easily deducible from (4.2).

LEMMA 2'. *Let $f(\theta)$ be of period 2π and of the class L^k, $k > 1$, and let*

$$f_k^*(\theta) = \max_{0<|h|\leqq\pi} \left| \frac{1}{h} \int_0^h |f(\theta + u)|^k du \right|^{1/k}.$$

(8) To simplify the notation we write $|t| \geqq \alpha\delta$ instead of $\pi \geqq |t| \geqq \alpha\delta$. Should it happen (for small ρ) that $\alpha\delta > \pi$, such values of ρ should be disregarded in the argument.

(9) See Hardy and Littlewood [2], or *TS*, p. 244.

(10) Hardy and Littlewood [3].

Then

$$(4.3)\qquad |f(\rho, \theta + u)| \leqq Cf_k^*(\theta)(1 + |u|/\delta)^{1/k}.$$

To deduce (4.3) from (4.2) it is enough to note that, by Jensen's inequality,

$$\left\{\frac{1}{\pi}\int_0^{2\pi} |f(\theta + t)| P(\rho, t)dt\right\}^k \leqq \frac{1}{\pi}\int_0^{2\pi} |f(\theta + t)|^k P(\rho, t)dt,$$

so that $|f(\rho, \theta)|^k$ is majorized by the Poisson integral of the function $|f(\theta)|^k$.

It is useful to observe that, for $u \geqq \alpha\delta$, the inequality (4.3) implies

$$|f(\rho, \theta + u)| \leqq Cf_k^*(\theta)(|u|/\delta)^{1/k}.$$

LEMMA 3. *Suppose that $\phi(z)$ is regular in $|z| < 1$, and belongs to the class H^λ, where $\lambda \geqq 2$. Let $\sigma > 1$, then*[11]

$$(4.4)\qquad \left\{\int_0^{2\pi} G_\sigma^\lambda(\theta)d\theta\right\}^{1/\lambda} \leqq C_{\lambda,\sigma}\left\{\int_0^{2\pi} |\phi(e^{i\theta})|^\lambda d\theta\right\}^{1/\lambda}.$$

For let μ be a number such that

$$\frac{1}{\lambda/2} + \frac{1}{\mu} = 1,$$

and let $\xi(\theta)$ be any positive function satisfying the inequality $\{\int_0^{2\pi} \xi^\mu d\theta\}^{1/\mu} \leqq 1$. Then

$$(4.5)\qquad \begin{aligned} \left\{\int_0^{2\pi} G_\sigma^\lambda(\theta)d\theta\right\}^{1/\lambda} &= \left\{\left[\int_0^{2\pi} (G_\sigma^2)^{\lambda/2} d\theta\right]^{2/\lambda}\right\}^{1/2} \\ &= \max_\xi \left\{\int_0^{2\pi} G_\sigma^2(\theta)\xi(\theta)d\theta\right\}^{1/2}. \end{aligned}$$

Let $h_\delta(u)$ denote the characteristic function of the interval $\alpha\delta \leqq |u| \leqq \pi$. Then, for any fixed $\xi(\theta)$,

$$(4.6)\qquad \begin{aligned} \int_0^{2\pi} G_\sigma^2(\theta)\xi(\theta)d\theta &\leqq C\int_0^{2\pi} \xi(\theta)d\theta \int_0^1 \delta^\sigma d\rho \int_{|t|\geqq\alpha\delta} |\phi'(\rho e^{i(\theta+t)})|^2 |t|^{-\sigma}dt \\ &= C\int_0^{2\pi} \xi(\theta)d\theta \int_0^1 \delta^\sigma d\rho \int_0^{2\pi} |\phi'(\rho e^{it})|^2 \frac{h_\delta(t-\theta)}{|t-\theta|^\sigma} dt \\ &= C\int_0^1 \delta d\rho \int_0^{2\pi} |\phi'(\rho e^{it})|^2 dt \delta^{\sigma-1} \int_0^{2\pi} \xi(\theta) \frac{h_\delta(t-\theta)}{|t-\theta|^\sigma} d\theta. \end{aligned}$$

---

(11) The proof given below is a modification of a proof given in Marcinkiewicz and Zygmund [7, p. 475]. This is an opportunity to correct a minor slip there. The sentence "It should be observed · · · ," on p. 478, line 7, should be deleted. This does not affect the argument.

The function $\xi^*(\theta)$ (cf. (4.1)) satisfies the inequality

$$\left(\int_0^{2\pi} \xi^{*\mu} d\theta\right)^{1/\mu} \leqq B_\mu \left(\int_0^{2\pi} \xi^\mu d\theta\right)^{1/\mu} \leqq B_\mu, \tag{4.7}$$

with $B_\mu \leqq 2\mu/(\mu-1)$. Let

$$\Xi(u) = \Xi(u; t) = \int_0^u [\xi(t+\theta) + \xi(t-\theta)] d\theta.$$

Integrating by parts the inner integral in the last part of (4.6), and rejecting negative terms we may write

$$\begin{aligned} \delta^{\sigma-1} \int_0^{2\pi} & \xi(\theta) h_\delta(\theta - t) \, | \theta - t |^{-\sigma} d\theta \\ &= \delta^{\sigma-1} \int_{\alpha\delta}^{\pi} [\xi(t+u) + \xi(t-u)] u^{-\sigma} du \\ &= \delta^{\sigma-1} [\Xi(u) u^{-\sigma}]_{\alpha\delta}^{\pi} + \sigma \delta^{\sigma-1} \int_{\alpha\delta}^{\pi} \Xi(u) u^{-\sigma-1} du \\ &\leqq \delta^{\sigma-1} \cdot \Xi(\pi) \cdot \pi^{-1} \cdot \pi^{-\sigma+1} + 2\sigma\delta^{\sigma-1} \xi^*(t) \int_{\alpha\delta}^{\infty} u^{-\sigma} du \\ &\leqq \xi^*(t) \left[2\delta^{\sigma-1} + \frac{2\sigma}{\sigma - 1} \alpha^{-(\sigma-1)} \right] \leqq C_\sigma' \xi^*(t). \end{aligned} \tag{4.8}$$

If σ is restricted to the range $1 < \sigma \leqq 2$, the only case which will be needed later, then

$$C_\sigma' \leqq C/(\sigma - 1).$$

Thus (4.6) implies

$$\begin{aligned} \int_0^{2\pi} G_\sigma^2(\theta) \xi(\theta) d\theta &\leqq C_\sigma' \int_0^1 \delta d\rho \int_0^{2\pi} | \phi'(\rho e^{it}) |^2 \xi^*(t) dt \\ &= C_\sigma' \int_0^{2\pi} \xi^*(t) g^2(t) dt \\ &\leqq C_\sigma' \left(\int_0^{2\pi} \xi^{*\mu} dt \right)^{1/\mu} \cdot \left(\int_0^{2\pi} g^\lambda dt \right)^{2/\lambda} \\ &\leqq C_\sigma' B_\mu \cdot A_\lambda^2 \left(\int_0^{2\pi} | f |^\lambda d\theta \right)^{2/\lambda} \leqq C_\sigma' B_\mu A_\lambda^2 \left(\int_0^{2\pi} | \phi(e^{i\theta}) |^\lambda d\theta \right)^{2/\lambda} \end{aligned}$$

(cf. (4.7) and (1.3)). From this and (4.5) the inequality (4.4) follows at once.

In particular, if $\sigma = 2$, we get (3.5) for $r \geqq 2$.

In order to prove (3.4) in the case $r \geqq 2$, we write[12]

$$S^2(\theta) = \iint_{\Omega_\theta} |\phi'|^2 d\omega = \iint_{|z|\leqq 1/2} |\phi'|^2 d\omega + \int_{1/2}^{1} d\rho \int_{|t|\leqq \alpha\delta} |\phi'(\rho e^{i(\theta+t)})|^2 dt$$
$$= U^2(\theta) + V^2(\theta),$$

say. Here

$$U(\theta) \leqq (\pi/4)^{1/2} \max_{|z|\leqq 1/2} \left| \frac{1}{2\pi i} \int_{|w|=3/4} \frac{\phi(w)}{(w-z)^2} dw \right|$$
$$\leqq C \int_0^{2\pi} |\phi(3e^{it}/4)| dt \leqq C \int_0^{2\pi} |\phi(e^{it})| dt \leqq C\left(\int_0^{2\pi} |\phi(e^{it})|^r dt \right)^{1/2}.$$

If $h^\delta(u)$ denotes the characteristic function of the interval $(-\alpha\delta, \alpha\delta)$, then

$$V^2(\theta) = \int_{1/2}^{1} d\rho \int_0^{2\pi} |\phi'(\rho e^{i(\theta+u)})|^2 h^\delta(u) du,$$

and arguing as in the proof of Lemma 3 (the argument even simplifies now, for unlike in (4.8) no integration by parts is necessary) we get (3.4) for $r \geqq 2$.

It remains to prove (3.4) and (3.5) for $1 < r \leqq 2$.

5. The inequality (4.4) is proved for $\sigma > 1$ and for all $\lambda \geqq 2$.

Let $\phi(z)$ be regular in $|z| < 1$ and of the class H^μ, where

$$1 < \mu < 2.$$

We assume temporarily that $\phi(z)$ has no zeros in $|z| < 1$, and set

$$\phi = \psi^{2/\mu}, \tag{5.1}$$

so that $\phi^\mu = \psi^2$, and

$$\begin{gathered} \psi \in H^2, \\ \phi' = (2/\mu)\psi^{2/\mu - 1}\psi', \\ |\phi'|^2 \leqq 4|\psi|^{2(2/\mu-1)}|\psi'|^2. \end{gathered} \tag{5.2}$$

Moreover,

$$\delta^\sigma \int_{|t|\geqq \alpha\delta} |\phi'(\rho e^{i(\theta+t)})|^2 |t|^{-\sigma} dt \leqq 4\delta^\sigma \int_{|t|\geqq\alpha\delta} |\psi|^{2(2/\mu-1)} |\psi'|^2 |t|^{-\sigma} dt. \tag{5.3}$$

Let k be any positive number less than 2, and let

$$M_k(\theta) = \max_{0<|h|\leqq\pi} \left| \frac{1}{h} \int_0^h |\psi(e^{i(\theta+u)})|^k du \right|^{1/k}. \tag{5.4}$$

---

[12] The α here is different from the α used previously.

It follows easily from Lemma 2 that

$$\left\{\int_0^{2\pi} M_k^2(\theta)d\theta\right\}^{1/2} \leqq B_k'\left\{\int_0^{2\pi} |\psi(e^{i\theta})|^2 d\theta\right\}^{1/2} \tag{5.5}$$

with $B_k' \leqq (4/(2-k))^{1/k}$.

From Lemma 2′, and taking into account that $\psi(\rho e^{it})$ is the Poisson integral of $\psi(e^{it})$, we see that the right-hand side of (5.3) does not exceed

$$C\delta^{\sigma} M_k^{2(2/\mu-1)}(\theta)\int_{|t|\geqq\alpha\delta}(1+|t|/\delta)^{(2/k)(2/\mu-1)}|\psi'|^2|t|^{-\sigma}dt$$
$$\leqq CM_k^{2(2/\mu-1)}(\theta)\delta^{\sigma'}\int_{|t|\geqq\alpha\delta}|\psi'|^2|t|^{-\sigma'}dt,$$

where

$$\sigma' = \sigma - (2/k)(2/\mu - 1). \tag{5.6}$$

Hence, integrating both sides of (5.3) over the interval $0\leqq\rho<1$,

$$\int_0^1 \delta^{\sigma}d\rho\int_{|t|\geqq\alpha\delta}|\phi'(\rho e^{i(\theta+t)})|^2|t|^{-\sigma}dt$$
$$\leqq CM_k^{2(2/\mu-1)}\theta\int_0^1\delta^{\sigma'}d\rho\int_{|t|\geqq\alpha\delta}|\psi'(\rho e^{i(\theta+t)})|^2|t|^{-\sigma'}dt,$$

which may be written $G_\sigma(\theta;\phi)\leqq CM_k^{2/\mu-1}(\theta)G_{\sigma'}(\theta;\psi)$ or

$$G_\sigma^\mu(\theta;\phi) \leqq CM_k^{2-\mu}(\theta)G_{\sigma'}^\mu(\theta;\psi). \tag{5.7}$$

So far the value of k has not been specified, except that it is to be less than 2. If we wish to apply Lemma 3 to $G_{\sigma'}$, we have to assume that $\sigma'>1$. Let us take $k=2/\mu$, $\sigma=2$ in (5.6). This gives

$$\sigma' = \mu > 1.$$

Let us now integrate both sides of (5.7) over the interval $0\leqq\theta\leqq 2\pi$, and let us apply Hölder's inequality to the right-hand side. We get (cf. (4.4) and (5.5))

$$\int_0^{2\pi} G_2^\mu(\theta;\phi)d\theta \leqq C\left\{\int_0^{2\pi} M_k^2(\theta)d\theta\right\}^{(2-\mu)/2}\cdot\left\{\int_0^{2\pi} G_\mu^2(\theta;\psi)d\theta\right\}^{\mu/2}$$
$$\leqq C(B_k')^{(2-\mu)}\left\{\int_0^{2\pi}|\psi(e^{i\theta})|^2d\theta\right\}^{(2-\mu)/2}$$
$$\cdot C_{2,\mu}^\mu\left\{\int_0^{2\pi}|\psi(e^{i\theta})|^2d\theta\right\}^{\mu/2}$$
$$= D_\mu\int_0^{2\pi}|\psi(e^{i\theta})|^2d\theta = D_\mu\int_0^{2\pi}|\phi(e^{i\theta})|^\mu d\theta.$$

Comparing the extreme terms of these inequalities, and taking account of (3.7), we deduce the inequality (3.5) for

$$1 < r(= \mu) < 2.$$

The proof of (3.4) in the case $1<r<2$ has some features in common with the proof of (3.5), but is much simpler. We consider the decomposition $S^2 = U^2 + V^2$ of section 4. The argument used for U holds true when $1<r\leqq 2$. In order to estimate V, we assume that $\phi(z)$ has no zeros inside the unit circle, and set $\phi = \psi^{2/\mu}$. Then (cf. (5.2))

$$V(\theta) = \left\{ \int_{1/2}^{1} d\rho \int_{|t|\leqq \alpha\delta} |\phi'(\rho e^{i(\theta+t)})|^2 dt \right\}^{1/2}$$

$$\leqq C\{M_1(\theta)\}^{2/\mu-1} \left\{ \int_{1/2}^{1} d\rho \int_{|t|\leqq \alpha\delta} |\psi'(\rho e^{i(\theta+t)})|^2 dt \right\}^{1/2},$$

where $M_1(\theta)$ is defined by (5.4) with $k=1$ (cf. also Lemma 2). Applying Hölder's inequality, we see that

$$\int_0^{2\pi} V^\mu d\theta \leqq C \left\{ \int_0^{2\pi} M_1^2 d\theta \right\}^{(2-\mu)/2} \left\{ \int_0^{2\pi} S^2(\theta; \psi) d\theta \right\}^{\mu/2}$$

$$\leqq D_\mu \int_0^{2\pi} |\psi(e^{i\theta})|^2 d\theta = D_\mu \int_0^{2\pi} |\phi(e^{i\theta})|^\mu d\theta.$$

Thus the inequalities (3.4) and (3.5), and so also the inequality

$$(5.8) \qquad \left\{ \int_0^{2\pi} g^{*r}(\theta) d\theta \right\}^{1/r} \leqq A_r^* \left\{ \int_0^{2\pi} |\phi(e^{i\theta})|^r d\theta \right\}^{1/r},$$

are established for $1<r\leqq 2$, provided that the function $\phi(z)$ does not vanish inside the unit circle. In order to get rid of this restriction we apply the familiar procedure. It is well known that if $\phi(z) \in H^r$, $r>0$, then $\phi = \phi_1 + \phi_2$, where both ϕ_1 and ϕ_2 are regular for $|z| < 1$, have no zeros there and satisfy the inequalities([13])

$$|\phi_1| \leqq |\phi|, \qquad |\phi_2| \leqq 2|\phi|.$$

By Minkowski's inequality,

$$f^*(\theta; \phi) \leqq f^*(\theta; \phi_1) + f^*(\theta; \phi_2).$$

Hence

$$\left\{ \int_0^{2\pi} g^{*r}(\theta; \phi) d\theta \right\}^{1/r} \leqq \left\{ \int_0^{2\pi} g^{*r}(\theta; \phi_1) d\theta \right\}^{1/r} + \left\{ \int_0^{2\pi} g^{*r}(\theta; \phi_2) d\theta \right\}^{1/r}$$

$$\leqq 3A_r^* \left\{ \int_0^{2\pi} |\phi|^r d\theta \right\}^{1/r},$$

and this completes the proof of the theorem.

([13]) Hardy and Littlewood [4].

§2. On an integral of Marcinkiewicz

1. Let $f(\theta)$ be an integrable function of period 2π, and let $f(\rho, \theta)$ denote the Poisson integral of $f(\theta)$. By $\phi(z)=\phi(z;f)$ we shall denote the analytic function whose real part is $f(\rho, \theta)$ and whose imaginary part vanishes for $z=0$.

Littlewood and Paley([14]) introduced the function

$$g(\theta) = g(\theta; f) = \left(\int_0^1 (1-\rho)\,|\phi'(\rho e^{i\theta})|^2 d\rho\right)^{1/2}.$$

They showed that, if $f\in L^r$, where $r>1$, then

$$(1.1)\quad A_r\left\{\int_0^{2\pi} |f(\theta)|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} g^r(\theta)d\theta\right\}^{1/r} \leqq A_r\left\{\int_0^{2\pi} |f(\theta)|^r d\theta\right\}^{1/r}$$

provided that in the first of these inequalities we assume that

$$(1.2)\qquad \phi(0) = 0.$$

If

$$1 < p \leqq 2 \leqq q < +\infty, \qquad r > 1,$$

and

$$g_r(\theta) = \left\{\int_0^1 (1-\rho)^{r-1}\,|\phi'(\rho e^{i\theta})|^r d\rho\right\}^{1/r},$$

so that $g_2(\theta)=g(\theta)$, then also([15])

$$(1.3)\qquad \left\{\int_0^{2\pi} g_q^q(\theta)d\theta\right\}^{1/q} \leqq A_q\left\{\int_0^{2\pi} |f(\theta)|^q d\theta\right\}^{1/q},$$

$$(1.4)\qquad \left\{\int_0^{2\pi} |f(\theta)|^p d\theta\right\}^{1/p} \leqq A_p\left\{\int_0^{2\pi} g_p^p(\theta)d\theta\right\}^{1/p}.$$

Let, for a moment, $\phi(z)$ denote *any* analytic function regular in $|z|<1$ (so that the real part of $\phi(z)$ need not be a Poisson integral). Let us assume that there is a set E of points θ such that $\phi(z)$ tends to a finite limit as $z\rightarrow e^{i\theta}$, $\theta\in E$, along any nontangential path. It has been shown that, under these circumstances, the function $g(\theta)$ is finite for almost every $\theta\in E$([16]).

It is easy to see that for every $\theta\in E$,

$$\phi'(\rho e^{i\theta}) = o(1-\rho)^{-1} \quad \text{as} \quad \rho\rightarrow 1.$$

This follows from the Cauchy formula

([14]) Littlewood and Paley [5].
([15]) See Marcinkiewicz and Zygmund [7].
([16]) Loc. cit.

$$\phi'(\rho e^{i\theta}) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\phi(\zeta)}{(\zeta - \rho e^{i\theta})^2}\,d\zeta = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\phi(\zeta) - \phi(\rho e^{i\theta})}{(\zeta - \rho e^{i\theta})^2}\,d\zeta,$$

where Γ_ρ denotes the circle with center at the point $\rho e^{i\theta}$ and of radius $(1-\rho)/2$. Thus, for every $\theta \in E$, the existence of $g(\theta)$ implies that of

$$g_q(\theta) = \left\{\int_0^1 (1-\rho)\,|\,\phi'(\rho e^{i\theta})\,|^2 \cdot |\,(1-\rho)\phi'(\rho e^{i\theta})\,|^{q-2} d\rho\right\}^{1/q}.$$

On the other hand, the function $g_p(\theta)$ may be always infinite even if the real part of $\phi(z)$ is the Poisson integral of a continuous function(17).

It is natural to look for functions analogous to $g(\theta)$ but defined without entering the interior of the unit circle. One might for example consider the function

$$\begin{aligned}\nu(\theta) &= \left\{\int_0^\pi \frac{[f(\theta+t) - f(\theta-t)]^2}{t}\,dt\right\}^{1/2}\\ &= \left\{4\int_0^\pi t\left[\frac{f(\theta+t) - f(\theta-t)}{2t}\right]^2 dt\right\}^{1/2},\end{aligned}$$

whose definition has a certain similarity to that of $g(\theta)$. It, however, turns out that the function $\nu(\theta)$ does not have the expected properties. For example, $\nu(\theta)$ may be everywhere infinite even if $f(\theta)$ is everywhere continuous(18).

Marcinkiewicz(19) had the right idea of introducing the function

$$\begin{aligned}\mu(\theta) = \mu(\theta; f) &= \left\{\int_0^\pi \frac{[F(\theta+t) + F(\theta-t) - 2F(\theta)]^2}{t^3}\,dt\right\}^{1/2}\\ &= \left\{\int_0^\pi t\left[\frac{F(\theta+t) + F(\theta-t) - 2F(\theta)}{t^2}\right]^2 dt\right\}^{1/2}\end{aligned} \tag{1.5}$$

where $F(\theta)$ is the integral of f,

$$F(\theta) = C + \int_0^\theta f(u)\,du.$$

More generally, he considers the functions

(17) This result was not proved by Littlewood and Paley and is stated here for the sake of completeness only. The corresponding result for the function μ_r (see below, property (d) of that function) was proved by Marcinkiewicz [6] and his argument may, without essential modification, be extended to the function g_r.

(18) See, for example, *TS*, p. 77, where a similar result is proved for the integral $\int_0^\pi |f(x+t) - f(x-t)|\,t^{-1}dt$. The same idea applies to the case discussed in the text.

(19) Marcinkiewicz [6]. For a generalization of the result, see Zygmund [11].

$$\mu_r(\theta) = \left\{\int_0^\pi \frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^r}{t^{r+1}}dt\right\}^{1/r} \tag{1.6}$$
$$= \left\{\int_0^\pi t^{r-1}\left|\frac{F(\theta+t)+F(\theta-t)-2F(\theta)}{t^2}\right|^r dt\right\}^{1/r},$$

so that $\mu_2(\theta)=\mu(\theta)$. He proves the following facts which are clearly analogues of the corresponding properties of $g(\theta)$.

(a) $$\left\{\int_0^{2\pi}\mu_q^q(\theta)d\theta\right\}^{1/q} \leqq A_q\left\{\int_0^{2\pi}|f(\theta)|^q d\theta\right\}^{1/q}, \qquad q \geqq 2,$$

(b) $$\left\{\int_0^{2\pi}|f(\theta)|^p d\theta\right\}^{1/p} \leqq A_p\left\{\int_0^{2\pi}\mu_p^p(\theta)d\theta\right\}^{1/p}, \qquad 1 < p \leqq 2.$$

(c) If $F(\theta)$ is the most general periodic function of the class L^2 (so that F need not be an integral) which has a finite derivative $F'(\theta)$ at every point θ of a set E, then

$$\mu(\theta) < +\infty$$

almost everywhere in E. In particular (since $[F(\theta+t)+F(\theta-t)-2F(\theta)]/t$ tends to 0 with t, for every $\theta\in E$),

$$\mu_q(\theta) < +\infty$$

almost everywhere in E.

(d) Let $1<p\leqq 2$. There exists a continuous function f of period 2π such that if F is the integral of f, the function $\mu_p(\theta)$ is infinite for almost every value of θ.

Marcinkiewicz also raises the question whether the function satisfies inequalities similar to (1.1), that is to say whether

$$B_r\left\{\int_0^{2\pi}|f(\theta)|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi}\mu^r(\theta)d\theta\right\}^{1/r} \tag{1.7}$$
$$\leqq B_r\left\{\int_0^{2\pi}|f(\theta)|^r d\theta\right\}^{1/r},$$

and showing that this is really so is the main purpose of the present note. The theorem may be stated as follows.

THEOREM 1. *Let $f(\theta)$ be of period 2π and of the class L^r, where $r>1$. Let $F(\theta)$ denote the integral of f. Then we have the inequalities* (1.7), *assuming in the case of the first of these inequalities that the constant term of the Fourier series of f is equal to* 0.

As Marcinkiewicz himself foresaw it[20], the proof of Theorem 1 is not

---

[20] Loc. cit.

quite easy. Besides the function $g(\theta)$ we shall have to use some other functions. Let us consider them in succession.

The first of them is

$$h(\theta) = h(\theta; f) = \left\{ \int_0^1 \rho^{-2}(1-\rho) \mid f_\theta(\rho, \theta) \mid^2 d\rho \right\}^{1/2}$$

and it obviously satisfies the inequality

$$h(\theta) \leqq g(\theta).$$

We shall require the following property of the function $h(\theta)$.

LEMMA 1. *Suppose that the constant term of the Fourier series of the function $f \in L^r$, $r > 1$, is equal to* 0. *Then*

$$\left\{ \int_0^{2\pi} \mid f(\theta) \mid^r d\theta \right\}^{1/r} \leqq A_r \left\{ \int_0^{2\pi} h^r(\theta) d\theta \right\}^{1/r}.$$

In view of the inequality $h \leqq g$, Lemma 1 is a slight strengthening of the first inequality in (1.1). It must however be added that, except for a minor modification, the proof given here of Lemma 1 is merely a repetition of the Littlewood-Paley proof of the first inequality (1.1).

Another function we shall have to use for the proof of Theorem 1 is the Littlewood-Paley function $g^*(\theta)$. It is defined by the formula

$$g^*(\theta) = g^*(\theta; f) = \left\{ \frac{1}{\pi} \int_0^{2\pi} (1-\rho) d\rho \int_0^1 \mid \phi'(\rho e^{i(\theta+u)}) \mid^2 P(\rho, u) du \right\}^{1/2},$$

where

$$P(\rho, u) = \frac{1}{2} \cdot \frac{1-\rho^2}{1 - 2\rho \cos u + \rho^2}$$

is the Poisson kernel.

LEMMA 2. *For every $r > 1$, the function g^* satisfies the inequality*

$$\left\{ \int_0^{2\pi} g^{*r}(\theta) d\theta \right\}^{1/r} \leqq A_r^* \left\{ \int_0^{2\pi} \mid f(\theta) \mid^r d\theta \right\}^{1/r}.$$

In the case $r = 2, 4, 6, \cdots$ the lemma was established by Littlewood and Paley[21]. For the general $r > 1$ the proof will be found in §1 of this paper.

There is one more function which we shall consider. Let $0 < \eta < 1$ be a fixed number and let $\Omega_\theta = \Omega_{\eta,\theta}$ be the domain limited by the tangents from the point $e^{i\theta}$ to the circle $|z| = \eta$ and by the more distant arc of that circle. If $\phi(z)$ is any function regular in $|z| < 1$, we shall consider the expression

[21] Littlewood and Paley [5].

$$S(\theta) = S_\eta(\theta;\phi) = \left\{ \iint_{\Omega_\theta} |\phi'(\rho e^{i(\theta+u)})|^2 \rho d\rho du \right\}^{1/2},$$

whose square represents the area of the image $w=\phi(z)$ of the domain Ω_θ. If the real part of $\phi(z)$ is the Poisson integral of a function $f(\theta)$, we may write $S(\theta;f)$ instead of $S(\theta;\phi)$.

LEMMA 3. *For every* $r>1$,

$$\left\{ \int_0^{2\pi} S^r(\theta;f)d\theta \right\}^{1/r} \leqq C_r \left\{ \int_0^{2\pi} |f(\theta)|^r d\theta \right\}^{1/r}$$

This inequality is also known(22). It is easy to see that

$$S_\eta(\theta;f) \leqq C_\eta^* g^*(\theta;f), \tag{1.8}$$

so that Lemma 3 is a consequence of Lemma 2, but a direct proof of Lemma 3 is simpler (23).

2. The first inequality in (1.7) is an immediate consequence of Lemma 1 and of the following lemma.

LEMMA 4. *For every* $r>1$, $h(\theta;f) \leqq C\mu(\theta;f)$.

Let us begin by proving Lemma 4.

Let

$$f(\theta) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta), \tag{2.1}$$

so that

$$F(\theta) = C + \frac{a_0\theta}{2} + \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)/\nu. \tag{2.2}$$

Without loss of generality we may assume that $a_0=0$, since neither $h(\theta)$ nor $\mu(\theta)$ depend on a_0. Hence the function $F(\theta)$ is periodic, and its Poisson integral $F(\rho,\theta)$ is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(u)P(\rho,\theta-u)du.$$

The Poisson integral of the function $f(\theta)$ is thus

$$f(\rho,\theta) = \partial F(\rho,\theta)/\partial\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} F(u)\frac{\partial}{\partial\theta}P(\rho,\theta-u)du.$$

Hence

(22) Marcinkiewicz and Zygmund [7].

(23) See the inequality (2.5) of §1.

$$f_\theta(\rho,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(u)\,\frac{\partial^2}{\partial\theta^2}P(\rho,\theta-u)du$$

$$(2.3)\qquad = \frac{1}{\pi}\int_{-\pi}^{\pi}\{F(\theta+u)+F(\theta-u)\}P_{uu}(\rho,u)du$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\{F(\theta+u)+F(\theta-u)-2F(\theta)\}P_{uu}(\rho,u)du,$$

since the function

$$(2.4)\qquad \begin{aligned} P_{uu} &= \{1/2+\rho\cos u+\rho^2\cos 2u+\cdots\}'' \\ &= -\rho\cos u - 2^2\rho^2\cos 2u - \cdots \end{aligned}$$

is even, and its integral extended over the interval $0\leqq u\leqq\pi$ is equal to 0.

Let

$$\delta = 1-\rho, \qquad 0<\delta\leqq 1.$$

We shall require the inequalities

$$(2.5)\qquad |P_{uu}| \leqq C\rho\delta^{-3} \qquad (0\leqq u\leqq\pi),$$

$$(2.6)\qquad |P_{uu}| \leqq C\rho\delta u^{-4} \qquad (\delta\leqq u\leqq\pi).$$

The inequality (2.5) follows from (2.4) and the fact that

$$\rho+2^2\rho^2+3^2\rho^3+\cdots = \rho(1+\rho)/(1-\rho)^3 \leqq 2\rho\delta^{-3}.$$

In order to prove (2.6), we write P_{uu} in the form

$$\rho(1-\rho^2)\cdot\frac{\{2\rho\sin^2 u+2(1+\rho^2)\sin^2(u/2)-(1-\rho)^2\}}{(1-2\rho\cos u+\rho^2)^3}\cdot$$

The denominator here exceeds Cu^6, and the absolute value of the numerator $\{\ \}$ does not exceed the larger of the numbers δ^2 and $2\rho\sin^2 u+2(1+\rho^2)\cdot\sin^2(u/2)\leqq 3u^2$. For $\delta\leqq u\leqq\pi$ the numerator does not numerically exceed $3u^2$, proving (2.6).

Let us now fix θ and set

$$F(\theta+u)+F(\theta-u)-2F(\theta) = \xi(u).$$

From (2.3), (2.5), and (2.6) it follows that

$$\delta\rho^{-2}|f_\theta(\rho,\theta)|^2 \leqq \delta\cdot C\delta^{-6}\left(\int_0^\delta|\xi(u)|du\right)^2+\delta\cdot C\delta^2\left(\int_\delta^\pi\frac{|\xi(u)|}{u^4}du\right)^2$$

$$\leqq C\delta^{-5}\left(\int_0^\delta\xi^2(u)du\right)\left(\int_0^\delta du\right)+C\delta^3\left(\int_\delta^\pi\frac{\xi^2(u)}{u^4}du\right)\left(\int_\delta^\pi\frac{du}{u^4}\right)$$

$$\leqq C\delta^{-4}\int_0^\delta\xi^2(u)du+C\int_\delta^\pi\frac{\xi^2(u)}{u^4}du = C\alpha(\delta)+C\beta(\delta),$$

say, so that

$$\begin{aligned}(2.7)\qquad \int_0^1 \rho^{-2\delta}\,|\,f_\theta(\rho,\theta)\,|^2 d\rho &= \int_0^1 \rho^{-2\delta}\,|\,f_\theta(\rho,\theta)\,|^2 d\delta \\ &\leqq C\int_0^1 \alpha(\delta)d\delta + C\int_0^1 \beta(\delta)d\delta.\end{aligned}$$

Since

$$\int_0^1 \alpha(\delta)d\delta \leqq \int_0^1 \delta^{-4}d\delta \int_0^\delta \xi^2(u)du \leqq \int_0^1 \xi^2(u)du \int_u^1 \delta^{-4}d\delta \leqq \int_0^\pi \frac{\xi^2(u)}{u^3}\,du,$$

and

$$\int_0^1 \beta(\delta)d\delta \leqq \int_0^1 d\delta \int_\delta^\pi \frac{\xi^2(u)}{u^4}\,du \leqq \int_0^\pi \frac{\xi^2(u)}{u^4}\,du \int_0^u d\delta \leqq \int_0^\pi \frac{\xi^2(u)}{u^3}\,du,$$

the right-hand side of (2.7) does not exceed $C\mu^2(\theta)$ and Lemma 4 is established.

3. Passing to the proof of Lemma 1, we may assume that the function $f(\rho, \theta)$ is harmonic in a circle of radius greater than 1, for otherwise instead of $f(\theta)$ we may consider the function $f_{R_0}(\theta)=f(R_0, \theta)$, $0<R_0<1$, whose Poisson integral $f(\rho R_0, \theta)$ is harmonic for $\rho<1/R_0$. If the inequality of Lemma 1 holds for the function $f_{R_0}(\theta)$, then making R_0 tend to 1 we obtain the inequality for the function f.

Let

$$M_r(f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |\,f(\theta)\,|^r d\theta\right)^{1/r},$$

$$(3.1)\qquad f_1(\theta) = |\,f\,|^{r-1}(\operatorname{sign} f) + cM_r^{r-1}(f),$$

where the constant c is so chosen that $\int_{-\pi}^{\pi} f_1 d\theta=0$. By Hölder's inequality, $|c|\leqq 1$, and by Minkowski's inequality,

$$M_{r'}(f_1) \leqq 2M_r^{r-1}(f) \qquad (1/r + 1/r' = 1).$$

Let $\phi(z)=\phi(z;f)=u+iv$, $\phi_1(z)=\phi(z;f_1)=u^*+iv^*$, $g_1(\theta)=g(\theta;f_1)$. Then f_1 is continuous in $|z|\leqq 1$, and so $(1-\rho)u_\rho^*\to 0$, uniformly in θ, as $\rho\to 1$. Now, if $0<R<1$,

$$\begin{aligned}\int_0^R (1-\rho)\frac{d^2}{d\rho^2}(uu^*)d\rho = [uu^*]_{\rho=R} + (1-R)\left[\frac{d}{d\rho}(uu^*)\right]_{\rho=R} \\ -\left[\frac{d}{d\rho}(uu^*)\right]_{\rho=0}.\end{aligned}$$

The last term on the right is 0, and the right-hand side converges, uniformly in θ, to $u(e^{i\theta})u^*(e^{i\theta})=f(\theta)f_1(\theta)$, as $R\to 1$. Hence

$$\int_0^{2\pi} |f|^r d\theta = \int_0^{2\pi} ff_1 d\theta \tag{3.2}$$
$$= \lim_{R\to 1} \int_0^R d\rho \int_{-\pi}^{\pi} (1-\rho)(u_{\rho\rho}u^* + uu^*_{\rho\rho} + 2u_\rho u^*_\rho)d\theta.$$

Now

$$u_{\rho\rho} = -u_\rho/\rho - u_{\theta\theta}/\rho^2,$$

so that

$$\int_0^{2\pi} u_{\rho\rho}u^* d\theta = -\frac{1}{\rho}\int_0^{2\pi} u_\rho u^* d\theta - \frac{1}{\rho^2}\int_0^{2\pi} u_{\theta\theta}u^* d\theta \tag{3.3}$$
$$= -\frac{1}{\rho}\int_0^{2\pi} u_\rho u^* d\theta + \frac{1}{\rho^2}\int_0^{2\pi} u_\theta u_\theta^* d\theta.$$

Similarly,

$$\int_0^{2\pi} uu^*_{\rho\rho} d\theta = -\frac{1}{\rho}\int_0^{2\pi} uu^*_\rho d\theta + \frac{1}{\rho^2}\int_0^{2\pi} u_\theta u^*_\theta d\theta. \tag{3.4}$$

In (3.4) we transform the first term on the right-hand side

$$-\frac{1}{\rho}\int_0^{2\pi} uu_\rho^* d\theta = -\frac{1}{\rho^2}\int_0^{2\pi} uv_\theta^* d\theta = \frac{1}{\rho^2}\int_0^{2\pi} u_\theta v^* d\theta,$$

so that

$$\int_0^{2\pi} uu^*_{\rho\rho} d\theta = \frac{1}{\rho^2}\int_0^{2\pi} u_\theta v^* d\theta + \frac{1}{\rho^2}\int_0^{2\pi} u_\theta u_\theta^* d\theta. \tag{3.5}$$

Similarly, for the first term on the right-hand side of (3.3) we get[24]

$$-\frac{1}{\rho}\int_0^{2\pi} u_\rho u^* d\theta = -\frac{1}{\rho^2}\int_0^{2\pi} v_\theta u^* d\theta = \frac{1}{\rho^2}\int_0^{2\pi} u_\theta v^* d\theta,$$

so that

$$\int_0^{2\pi} u_{\rho\rho}u^* d\theta = \frac{1}{\rho^2}\int_0^{2\pi} u_\theta v^* d\theta + \frac{1}{\rho^2}\int_0^{2\pi} u_\theta u_\theta^* d\theta. \tag{3.6}$$

[24] Let $F=U+iV$ and $F_1=U_1+iV_1$ be two functions regular in $|z|<1$, and let $F(0)=0$. Since the integral of $F(z)F_1(z)z^{-1}$ taken along the circle $|z|=\rho<1$ is 0, we get $\int_0^{2\pi} UU_1 d\theta = \int_0^{2\pi} VV_1 d\theta$, $\int_0^{2\pi} UV_1 d\theta = -\int_0^{2\pi} U_1 V d\theta$. Here we use the second of these formulas with $U+iV = u_\theta + iv_\theta$, $U_1+iV_1 = u^*+iv^*$. The first of these formulas will be used in (3.7).

We also note that

$$\int_0^{2\pi} u_\rho u_\rho^* d\theta = \frac{1}{\rho^2}\int_0^{2\pi} v_\theta v_\theta^* d\theta = \frac{1}{\rho^2}\int_0^{2\pi} u_\theta u_\theta^* d\theta. \tag{3.7}$$

From (3.2), (3.5), (3.6), (3.7) and from the inequalities

$$|v^*| \leqq |\phi_1|, \qquad |u_\theta^*| \leqq \rho|\phi_1'|, \qquad |v_\theta^*| \leqq \rho|\phi_1'|$$

we deduce

$$\begin{aligned}
\int_0^{2\pi} |f|^r d\theta &\leqq 4\int_0^{2\pi}\int_0^1 \rho^{-1}(1-\rho)|u_\theta||\phi_1'|d\rho d\theta \\
&\quad + 2\int_0^{2\pi}\int_0^1 \rho^{-2}(1-\rho)|u_\theta\phi_1|d\rho d\theta \\
&\leqq 4\int_0^{2\pi} d\theta\left(\int_0^1 \rho^{-2}(1-\rho)u_\theta^2 d\rho\right)^{1/2} \\
&\quad \cdot\left(\int_0^1 (1-\rho)|\phi_1'|^2 d\rho\right)^{1/2} \\
&\quad + 2\int_0^{2\pi}\Phi_1(\theta)\left(\int_0^1 (1-\rho)\rho^{-2}u_\theta^2 d\rho\right)^{1/2} d\theta \\
&= 4\int_0^{2\pi} h(\theta)g_1(\theta)d\theta + 2\int_0^{2\pi} h(\theta)\Phi_1(\theta)d\theta,
\end{aligned} \tag{3.8}$$

where $\Phi_1(\theta) = \max |\phi_1(\rho e^{i\theta})/\rho e^{i\theta}|$ for $0 \leqq \rho < 1$.

By the theorem of Hardy and Littlewood,

$$\begin{aligned}
\left\{\int_0^{2\pi} \Phi_1^{r'}(\theta)d\theta\right\}^{1/r'} &\leqq C_{r'}\left\{\int_0^{2\pi} |\phi_1(e^{i\theta})|^{r'} d\theta\right\}^{1/r'} \\
&\leqq D_{r'}\left\{\int_0^{2\pi} |f_1(\theta)|^{r'} d\theta\right\}^{1/r'},
\end{aligned}$$

and by the second inequality (1.1) applied to f_1,

$$\left\{\int_0^{2\pi} g_1^{r'} d\theta\right\}^{1/r'} \leqq A_{r'}\left\{\int_0^{2\pi} |f_1|^{r'} d\theta\right\}^{1/r'}.$$

Hence the right-hand side of (3.8) does not exceed

$$\begin{aligned}
D_r^*&\left\{\int_0^{2\pi} h^r d\theta\right\}^{1/r}\left\{\int_0^{2\pi} |f_1|^{r'} d\theta\right\}^{1/r'} \\
&\leqq D_r^*\left\{\int_0^{2\pi} h^r d\theta\right\}^{1/r}\left\{\int_0^{2\pi} |f|^r d\theta\right\}^{(r-1)/r}.
\end{aligned}$$

Comparing this with the left-hand side of (3.8), we get

$$\left\{\int_0^{2\pi} |f|^r d\theta\right\}^{1/r} \leqq D_r^* \left\{\int_0^{2\pi} h^r d\theta\right\}^{1/r}.$$

This completes the proof of Lemma 1, and so also of the first inequality in (1.7).

4. The second inequality (1.7) will be a consequence of Lemma 2 and of the following lemma.

LEMMA 5. *For every $r>1$,*

$$\mu(\theta) \leqq Cg^*(\theta). \tag{4.1}$$

We fix θ and set

$$\begin{aligned} F_1(\rho, u) &= F(\rho, \theta+u) + F(\rho, \theta-u) - 2F(\rho, \theta), \\ f_1(\rho, u) &= f(\rho, \theta+u) - f(\rho, \theta-u), \\ \rho_u &= 1 - (u/2\pi), \end{aligned}$$

so that

$$f_1(\rho, u) = \partial F_1(\rho, u)/\partial u,$$

and the interval $0 \leqq u \leqq \pi$ is mapped onto $1/2 \leqq \rho \leqq 1$. We may write

$$F_1(t) \equiv F_1(1, t) = \{F_1(1, t) - F_1(\rho, t)\} + \{F_1(\rho, t) - F_1(\rho, 0)\} = V + W,$$

say, where $\rho = \rho_t$. Now,

$$\begin{aligned} W^2 &\leqq \left(\int_0^t \left|\frac{\partial F_1(\rho, u)}{\partial u}\right| du\right)^2 \leqq t\int_0^t \left|\frac{\partial F_1}{\partial u}\right|^2 du = t\int_0^t |f_1(\rho, u)|^2 du \\ &= t\int_0^t du\left(\int_{-u}^{u} \frac{\partial f(\rho, \theta+v)}{\partial v} dv\right)^2 \leqq t\int_0^t \left\{2u\int_{-u}^{u}\left|\frac{\partial f(\rho, \theta+v)}{\partial v}\right|^2 dv\right\} du \\ &= t\int_{-t}^{t}\left|\frac{\partial f(\rho, \theta+v)}{\partial v}\right|^2 dv \left\{\int_{|v|}^{t} 2u\,du\right\} \leqq t^3 \int_{-t}^{+t}\left|\frac{\partial f(\rho, \theta+v)}{\partial v}\right|^2 dv. \end{aligned}$$

Since $\rho \leqq 1$,

$$\begin{aligned} \int_0^{\pi} W^2(t) t^{-3} dt &\leqq \int_0^{\pi} dt \int_{-t}^{+t} \left|\frac{\partial f(\rho, \theta+v)}{\rho\,\partial v}\right|^2 \rho^2 dv \\ &\leqq 2\pi \iint_{\Omega_{\eta_0, \theta}} |\phi'(\rho e^{iu})|^2 \rho\, d\rho\, du = 2\pi S_{\eta_0}^2(\theta) \leqq Cg^{*2}(\theta), \end{aligned} \tag{4.2}$$

where η_0 is a positive absolute constant less than 1, and $\Omega_{\eta_0,\theta}$ is the region defined in section 1.

In order to estimate V, we note that for every function $\lambda(x)$ having a continuous second derivative in a closed interval (a, b),

$$\lambda(b) = \lambda(a) + (b - a)\lambda'(a) + \int_a^b (b - x)\lambda''(x)dx.$$

Thus, for every t,

$$V = \lim_{R\to 1} \left\{ \int_{\rho_t}^R (R - \rho)[F_{\rho\rho}(\rho, \theta + t) + F_{\rho\rho}(\rho, \theta - t) - 2F_{\rho\rho}(\rho, \theta)]d\rho \right\}$$
$$+ (1 - \rho_t)\{F_\rho(\rho_t, \theta + t) + F_\rho(\rho_t, \theta - t) - F_\rho(\rho_t, \theta)\}.$$

Let a_0 $(=0)$, a_1, b_1, $\cdots$ be the Fourier coefficients of the function f, so that $f(\rho, \theta)$, $F(\rho, \theta)$, and the function $\tilde{f}(\rho, \theta)$ conjugate to $f(\rho, \theta)$ (and vanishing at the origin) are given by the formulas

$$f(\rho, \theta) = \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)\rho^\nu,$$

$$F(\rho, \theta) = \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)\nu^{-1}\rho^\nu,$$

$$\tilde{f}(\rho, \theta) = \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)\rho^\nu.$$

It follows that $F_\rho = \tilde{f}\rho^{-1}$; $F_{\rho\rho} = -(f_\theta + \tilde{f})\rho^{-2}$, and, consequently,

$$V = \lim_{R\to 1} \left\{ -\int_{\rho_t}^R (R - \rho)\rho^{-2}[f_\theta(\rho, \theta + t) + f_\theta(\rho, \theta - t) - 2f_\theta(\rho, \theta)]d\rho \right.$$
$$\left. - \int_{\rho_t}^R (R - \rho)\rho^{-2}[\tilde{f}(\rho, \theta + t) + \tilde{f}(\rho, \theta - t) - 2\tilde{f}(\rho, \theta)]d\rho \right\}$$
$$+ (1 - \rho_t)\rho_t^{-1}\{\tilde{f}(\rho_t, \theta + t) + \tilde{f}(\rho_t, \theta - t) - 2\tilde{f}(\rho_t, \theta)\}.$$

This implies the inequality

$$\begin{aligned} |V| \leqq & \int_{\rho_t}^1 (1 - \rho)\rho^{-2}| f_\theta(\rho, \theta + t) + f_\theta(\rho, \theta - t) - 2f_\theta(\rho, \theta) | d\rho \\ & + \int_{\rho_t}^1 (1 - \rho)\rho^{-2}| \tilde{f}(\rho, \theta + t) + \tilde{f}(\rho, \theta - t) - 2\tilde{f}(\rho, \theta) | d\rho \\ & + (1 - \rho_t)\rho_t^{-1} | \tilde{f}(\rho_t, \theta + t) + \tilde{f}(\rho_t, \theta - t) - 2\tilde{f}(\rho_t, \theta) | \\ = & V_1 + V_2 + V_3, \end{aligned} \tag{4.3}$$

say.

Let us first consider the term V_1. Since $\rho_t \geqq 1/2$,

$$V_1 \leqq 4\left\{ \int_{\rho_t}^{1} (1-\rho)\,|\,f_\theta(\rho_t, \theta+t)\,|\,d\rho + \int_{\rho_t}^{1} (1-\rho)\,|\,f_\theta(\rho_t, \theta-t)\,|\,d\rho \right.$$

$$\left. + \int_{\rho_t}^{1} (1-\rho)\,|\,f_\theta(\rho_t, \theta)\,|\,d\rho \right\} \tag{4.4}$$

$$= 4\{V_{11} + V_{12} + V_{13}\},$$

say, and

$$V_1^2 \leqq C(V_{11}^2 + V_{12}^2 + V_{13}^2). \tag{4.5}$$

Let us write

$$\delta = 1 - \rho, \qquad \delta_t = 1 - \rho_t = t/2\pi.$$

An application of Schwarz's inequality gives

$$V_{11}^2 \leqq (1-\rho_t) \int_{\rho_t}^{1} (1-\rho)^2 f_\theta^2(\rho, \theta+t)d\rho \leqq t \int_0^{\delta_t} \delta^2 f_\theta^2(\rho, \theta+t)d\delta,$$

so that

$$\int_0^\pi V_{12}^2 t^{-3} dt \leqq \int_0^\pi t^{-2} dt \int_0^{\delta_t} \delta^2 f_\theta^2(\rho_t, \theta+t)d\delta$$

$$\leqq C \iint_{\Omega'_{\eta_1,0}} \frac{\delta^2}{t^2}\,|\,\phi'(\rho e^{i(\theta+u)})\,|^2 \rho d\rho du,$$

where $0<\eta_1<1$ is an absolute constant, and $\Omega'_{\eta,u}$ is the region complementary to $\Omega_{\eta,u}$. The last integral does not exceed a fixed multiple of $g^{*2}(\theta)$ (cf. section 3 of §1). A similar inequality holds if V_{11} is replaced by V_{12}. Thus

$$\int_0^\pi (V_{11}^2 + V_{12}^2)t^{-3} dt \leqq Cg^{*2}(\theta). \tag{4.6}$$

Again,

$$V_{13}^2 \leqq \delta_t \int_0^{\delta_t} \delta^2 f_\theta^2(\rho, \theta)d\delta \leqq t \int_0^{\delta_t} \delta^2 f_\theta^2(\rho, \theta)d\delta,$$

so that

$$\int_0^\pi V_{13}^2 t^{-3} dt \leqq \int_0^\pi t^{-2} dt \int_0^{t/2\pi} \delta^2 f_\theta^2(\rho, \theta)d\delta \leqq \int_0^{1/2} \delta^2 f_\theta^2(\rho, \theta)d\delta \int_{2\pi\delta}^\infty t^{-2} dt$$

$$\leqq C \int_0^{1/2} \delta f_\theta^2(\rho, \theta)d\delta \leqq C \int_0^1 \delta\,|\,\phi'(\rho e^{i\theta})\,|^2 d\rho = Cg^2(\theta) \leqq Cg^{*2}(\theta)$$

(cf. the inequality (1.7) of §1). This, (4.6) and (4.4) imply

(4.7) $$\int_0^\pi V_1^2 t^{-3} dt \leqq C g^{*2}(\theta).$$

Passing to the expression V_3 we note that

$$V_3^2 \leqq \rho_t^{-2} \delta_t^2 \left(\int_{-t}^{+t} \left| \tilde{f}_\theta(\rho_t, \theta + u) \right| du \right)^2 \leqq C t^3 \int_{-t}^{t} \left| \phi'(\rho_t e^{i(\theta+u)}) \right|^2 du,$$

so that, as is easily seen (cf. (1.8)),

(4.8) $$\int_0^\pi V_3^2 t^{-3} dt \leqq C \int_0^\pi dt \int_{-t}^{t} \left| \phi'(\rho_t e^{i(\theta+u)}) \right|^2 du \leqq C S_{\eta_2}^2(\theta) \leqq C g^{*2}(\theta),$$

where $0 < \eta_2 < 1$. Let us now write, for $\rho_t \leqq \rho < 1$,

$$\tilde{f}(\rho, u) = \tilde{f}(\rho_t, u) + \int_{\rho_t}^{\rho} \tilde{f}_\rho(R, u) dR.$$

Then, obviously,

$$\begin{aligned} V_2 &\leqq 4 \left| \tilde{f}(\rho_t, \theta + t) + \tilde{f}(\rho_t, \theta - t) - 2\tilde{f}(\rho_t, \theta) \right| \int_{\rho_t}^{1} (1 - \rho) d\rho \\ &\quad + 4 \int_{\rho_t}^{1} (1 - \rho) d\rho \int_{\rho_t}^{\rho} \left| \tilde{f}_\rho(R, \theta + t) + \tilde{f}_\rho(R, \theta - t) - 2\tilde{f}_\rho(R, \theta) \right| dR \\ &= V_{21} + V_{22}, \end{aligned}$$

say. Here

$$V_{21} = 2(1 - \rho_t^2) \left| \tilde{f}(\rho_t, \theta + t) + \tilde{f}(\rho_t, \theta - t) - 2\tilde{f}(\rho_t, \theta) \right|,$$

and

$$\begin{aligned} V_{22} &= 2 \int_{\rho_t}^{1} (1 - R)^2 \left| \tilde{f}_\rho(R, \theta + t) + \tilde{f}_\rho(R, \theta - t) - 2\tilde{f}_\rho(R, \theta) \right| dR \\ &\leqq 2 \int_{\rho_t}^{1} (1 - \rho) \left| \tilde{f}_\rho(\rho, \theta + t) + \tilde{f}_\rho(\rho, \theta - t) - 2\tilde{f}_\rho(\rho, \theta) \right| d\rho. \end{aligned}$$

Since $1/2 \leqq \rho_t \leqq 1$, we see that V_{21} does not exceed a fixed multiple of V_3. Similarly, taking into account that $\tilde{f}_\rho = -f_\theta/\rho$, we notice that the expression V_{22} does not exceed a fixed multiple of V_1. Thus, on account of (4.7) and (4.8),

$$\int_0^\pi V_2^2 t^{-3} dt \leqq 4 \int_0^\pi (V_{21}^2 + V_{22}^2) t^{-3} dt \leqq C \int_0^\pi (V_1^2 + V_3^2) t^{-3} dt \leqq C g^{*2}(\theta)$$

and, consequently,

$$\int_0^\pi V^2 t^{-3} dt \leqq C \int_0^\pi (V_1^2 + V_2^2 + V_3^2) t^{-3} dt \leqq C g^{*2}(\theta).$$

Finally (cf. (4.2))

$$\mu^2(\theta) \leqq 4 \int_0^\pi (V^2 + W^2) t^{-3} dt \leqq C g^{*2}(\theta).$$

This completes the proof of Lemma 5, and so also of Theorem 1.

§3. On the absolute summability A of series

1. A series

$$c_0 + c_1 + \cdots + c_n + \cdots \tag{1.1}$$

is said to be absolutely summable A, or summable $|A|$, if the series

$$c_0 + c_1\rho + c_2\rho^2 + \cdots \tag{1.2}$$

converges for $0 \leqq \rho < 1$, and its sum, which we shall denote by $\phi(\rho)$, is of bounded variation over the interval $0 \leqq \rho < 1$.

Summability $|A|$ obviously implies summability A. If the series $c_0 + c_1 + \cdots$ converges absolutely, then it may be represented as a difference of two convergent series with non-negative terms. Correspondingly, the function $\phi(\rho)$ is a difference of two bounded, non-decreasing and non-negative functions. Thus $\phi(\rho)$ is of bounded variation, which shows that absolute convergence implies absolute summability A. The converse is obviously false.

In what follows, we shall denote by $V(f; a, b)$, or by $V(f; I)$, the absolute variation of the function f over the interval $I = (a, b)$, which may be closed or half-closed.

The series (1.1) will be called *lacunary* if its terms are all zero, except perhaps for the indices

$$n_0 = 0 < n_1 < n_2 < \cdots$$

satisfying a condition

$$n_{\nu+1}/n_\nu > q > 1, \qquad \nu = 0, 1, 2, \cdots.$$

The "high-indices" theorem of Hardy and Littlewood asserts that if a lacunary series is summable A, it is convergent. We shall prove the following parallel result.

THEOREM 1. *If the series* (1.1) *is lacunary and summable* $|A|$, *then it is absolutely convergent. Moreover*

$$\sum_{\nu=1}^{\infty} |c_\nu| \leqq A_q V(\phi; 0, 1). \tag{1.3}$$

The proof given below of Theorem 1 borrows the main idea from Ingham's proof of the high-indices theorem of Hardy and Littlewood([25]).

If we apply Theorem 1 to lacunary trigonometric series

$$\sum_{\nu=1}^{\infty} (a_\nu \cos n_\nu\theta + b_\nu \sin n_\nu\theta), \qquad n_{\nu+1}/n_\nu > q > 1, \tag{1.4}$$

we get as a corollary the following theorem.

THEOREM 2. *If the series* (1.4) *is summable* $|A|$ *for every point of a set E of positive measure, then the series*

$$\sum_{\nu=1}^{\infty} |a_\nu| + |b_\nu|$$

converges.

For from Theorem 1 we see that

$$\sum_{\nu=1}^{\infty} | a_\nu \cos n_\nu\theta + b_\nu \sin n_\nu\theta |$$

for every $\theta \in E$, and it is sufficient to apply the very well known theorem of Lusin and Denjoy concerning the absolute convergence of trigonometric series([26]).

It is well known that the harmonic function

$$\sum_{\nu=1}^{\infty} (a_\nu \cos n_\nu\theta + b_\nu \sin n_\nu\theta)\rho^{n_\nu}$$

tends to a finite limit along almost every radius, provided the sum $\Sigma(a_\nu^2+b_\nu^2)$ is finite. Theorem 2 shows that unless we assume that $\Sigma(|a_\nu|+|b_\nu|)< +\infty$, that convergence is almost always "bad."

Summability $|A|$ of the series (1.1) is equivalent to the finiteness of the integral

$$\int_0^1 | \phi'(\rho) | \, d\rho, \tag{1.5}$$

whose value represents the total variation of the function $\phi(\rho)$ over the interval $0 \leqq \rho < 1$.

This integral has some connection with the integrals considered in §2, for if we set

$$g_r(\theta) = g_r(\theta; \phi) = \left\{ \int_0^1 (1-\rho)^{r-1} | \phi'(\rho e^{i\theta}) |^r d\rho \right\}^{1/r},$$

([25]) Ingham [1]. Lemma 1 below is modeled on his argument.

([26]) See, for example, *TS*, p. 131.

where $\phi(z)$ is any function regular in $|z|<1$, then (1.5) is $g_1(0;\phi)$.

It has been proved([27]) that if $\phi(z)$, regular in $|z|<1$, has a nontangential limit

$$\phi(e^{i\theta}) = \lim_{z\to e^{i\theta}} \phi(z)$$

for every value of θ belonging to a set E of positive measure, then the function,

$$g(\theta) = g_2(\theta;\phi) = \left(\int_0^1 (1-\rho)\,|\,\phi'(\rho e^{i\theta})\,|^2 d\rho\right)^{1/2}$$

is finite for almost every $\theta\in E$. For every such θ, Schwarz's inequality gives

$$\int_0^\rho |\,\phi'(Re^{i\theta})\,|\,dR \leqq \left\{\int_0^\rho (1-R)\,|\,\phi'(Re^{i\theta})\,|^2 dR\right\}^{1/2}\left\{\int_0^\rho \frac{dR}{1-R}\right\}^{1/2}$$
$$= O\left(\log^{1/2}\frac{1}{1-\rho}\right)$$

and it is immediate that O may be replaced here by o. Hence, if $W(\rho,\theta) = W(\rho,\theta;\phi)$ denotes the total variation of the function $\phi(z)$ over the segment $z=Re^{i\theta}$, $0\leqq R\leqq\rho$, then

$$W(\rho,\theta) = o(\log^{1/2}(1/(1-\rho)))$$

for almost every point θ at which $\phi(z)$ has a nontangential limit. That this is the best possible result is shown by the following theorem.

THEOREM 3. *For every function* $\epsilon(\rho)$, $0\leqq\rho<1$, *positive and tending to* 0 *as* $\rho\to1$, *there is a regular function* $\phi(z)$, $|z|<1$, *of the class* H^2 (*and so having a nontangential limit almost everywhere*) *such that*

$$W(\rho,\theta) \neq O\{\epsilon(\rho)\log^{1/2}(1/(1-\rho))\}$$

for almost every θ.

Such a function may be constructed either as a lacunary series or by averaging power series whose coefficients have rather smooth absolute values but random signs. The latter method suggests the problem of the effect of a random change of the signs of the coefficients upon the behavior of the function $W(\rho,\theta)$. The following result gives an answer.

THEOREM 4. *For every* $0\leqq t\leqq 1$, *let*

$$\phi_t(z) = \sum_{\nu=0}^{\infty} c_\nu\psi_\nu(t)z^\nu$$

([27]) Marcinkiewicz and Zygmund [7].

where $\psi_0(t), \psi_1(t), \cdots$ *are Rademacher's functions. If the series*

$$\sum_{n=0}^{\infty}\left\{\sum_{\nu=2^n+1}^{2^{n+1}} |c_\nu|^2\right\}^{1/2} \tag{1.6}$$

converges, then for almost every t the expression $W(1, \theta; \phi_t)$ *is finite almost everywhere in* θ. *If the series* (1.6) *is divergent, then for almost every t the expression* $W(1, \theta; \phi_t)$ *is infinite almost everywhere in* θ.

2. Passing to the proof of Theorem 1, we make the substitution $\rho = e^{-s}$, which transforms the power series (1.2) into a Dirichlet series

$$\sum_{k=1}^{\infty} a_k e^{-\lambda_k s} = f(s), \qquad 0 < \lambda_1 < \lambda_2 < \cdots, \ \lambda_{k+1}/\lambda_k > q > 1. \tag{2.1}$$

(We replace n_k by λ_k, and write a_k instead of c_{n_k}; since both sides of (1.3) are independent of c_0, we may assume that $c_0 = 0$.) We are going to show that

$$\sum_{k=1}^{\infty} |a_k| \leqq A_q V(f; 0, +\infty) \tag{2.2}$$

and in the proof we shall not use the fact that the numbers λ_k are integers. Hence (2.2) represents a slight generalization of Theorem 1.

First of all we prove (2.2) for Dirichlet polynomials.

LEMMA 1. *If* $0<\lambda_1<\lambda_2<\cdots, \lambda_{k+1}/\lambda_k>q>1$, *then the total variation of the finite sum* $f(s)=\sum a_k e^{-\lambda_k s}$ *satisfies the inequality*

$$\sum |a_k| \leqq A_q V(f; 0, +\infty). \tag{2.3}$$

Proof. We note that if

$$P(s) = \sum p_l e^{-\mu_l s}, \qquad 0 < \mu_1 < \mu_2 < \cdots,$$

is a finite sum, then

$$F(s) \equiv \sum p_l f(\mu_l s) = \sum a_k P(\lambda_k s). \tag{2.4}$$

Let us set $p(s) = e^{-\alpha s} - e^{-\beta s}$, where $0<\alpha<\beta$, for example, $\alpha=1, \beta=2$. This function increases as s increases from 0 to a certain value $\sigma=\sigma(\alpha, \beta)$, and later on decreases as $s \to +\infty$. Moreover $p(0)=p(+\infty)=0$. Let us set

$$P(s) = \{p(\sigma s)/p(\sigma)\}^R,$$

where R is a positive integer to be determined later. Hence $P(1)=1$. The larger R is, the steeper is the peak at $s=1$. Obviously $\sum |p_l| = 2^R/p^R(\sigma)$.

Let I_k denote the interval $(q^{-1/2}\lambda_k^{-1}, q^{1/2}\lambda_k^{-1})$, so that no two I's overlap. The total variation of $P(\lambda_k s)$ over the interval I_k is the same as the variation of $P(s)$ over the interval $(q^{-1/2}, q^{1/2})$. If $m \neq k$, the variation of $P(\lambda_m s)$ over I_k is the same as the variation of $P(s)$ over the interval $(\lambda_m q^{-1/2}\lambda_k^{-1}, \lambda_m q^{1/2}\lambda_k^{-1})$.

Hence it does not exceed $P(q^{m-k-1/2})$ if $m>k$, and does not exceed $P(q^{m-k+1/2})$ if $m<k$.

From (2.4) we get at once

$$(2.5)\qquad |a_k|V(P(\lambda_k s); I_k) - \sum_{m\neq k}|a_m|V(P(\lambda_m s); I_k) \leqq V(F(s); I_k).$$

Let us take R so large that the variation of $P(s)$ over the interval $(q^{-1/2}, q^{1/2})$ exceeds 1. Hence, summing (2.5) with respect to k, we get

$$\sum_k |a_k| \{1 - [P(q^{-1/2}) + P(q^{-3/2}) + \cdots] - [P(q^{1/2}) + P(q^{3/2}) + \cdots]\} \leqq V(F; 0, \infty) \leqq 2^R p^{-R}(\sigma)V(f; 0, +\infty).$$

The function $P(s)$ is $O(s^R)$ for $s\to+0$, and is $O(e^{-sR})=O(s^{-R})$ for $s\to+\infty$. Hence, if we take R large enough, we may assume that

$$P(q^{-1/2}) + P(q^{-3/2}) + \cdots + P(q^{1/2}) + P(q^{3/2}) + \cdots < 1/2,$$

and this in connection with the previous inequality gives

$$(1/2)\sum|a_k| \leqq 2^R p^{-R}(\sigma)V(f; 0, +\infty).$$

The proof of Lemma 1 is thus complete. The above argument shows that the inequality (2.3) is valid if $V(f; 0, +\infty)$ on the right is replaced by $V(f; 0, q^{1/2}\lambda_1^{-1})$.

In order to prove Theorem 1, let I^δ denote the interval $(\delta, q^{1/2}\lambda_1^{-1})$, where $\delta>0$, and let $f_N(s)$ denote the partial sums of the infinite series (2.1). Since the latter series differentiated term by term converges uniformly over I^δ to the value $f'(s)$, we get that $V(f_N; I^\delta)\to V(f; I^\delta)$ as $N\to+\infty$, and so

$$V(f_N; I^\delta) \leqq V(f; I^\delta) + \epsilon \leqq V(f; 0, +\infty) + \epsilon$$

for $N>N_0(\epsilon, \delta)$. Hence, on account of Lemma 1,

$$\sum_{k=1}^{N}|a_k|e^{-\lambda_k\delta} \leqq V(f; 0, +\infty) + \epsilon.$$

First making $N\to+\infty$, then $\delta\to0$, and finally $\epsilon\to0$, we get (2.2).

Remark. If by A_q in (2.2) we mean the least value of that constant, then:

(i) *The constant A_q tends to $+\infty$ as $q\to1$. However,* $A_q\leqq\exp\{A/(q-1)^2\}$ *as $q\to1$.*

(ii) *The constant A_q tends to* 1 *as $q\to+\infty$.*

Part (i) easily follows from the proof of Lemma 1.

Part (ii) may also be obtained from the proof of Lemma 1 by taking α small, β large, and $R=1$. The details may be left to the reader.

3. Passing to the proof of Theorem 3, we may assume that $\epsilon(\rho)$ is a positive and decreasing function of ρ, and that $\epsilon(\rho)\log^{1/2}(1/(1-\rho))\to+\infty$. Let us consider any sequence a_n of, say, positive numbers such that $\sum a_n^2$ converges, and let

$$\phi(z) = \sum_{n=1}^{\infty} a_n z^{2^n}. \tag{3.1}$$

Let us suppose, moreover, that

$$\int_0^\rho |\phi'(Re^{i\theta})|\, dR = O(\epsilon(\rho) \log^{1/2} (1/(1-\rho))) \qquad (\rho \to 1) \tag{3.2}$$

in a set of θ of positive measure. We can then find a constant A and a set E of positive measure such that

$$\int_0^\rho |\phi'(Re^{i\theta})|\, dR \leqq A\epsilon(\rho) \log^{1/2} (1/(1-\rho))$$

for $\rho > \rho_0$, $\theta \subset E$. If $\psi = \psi_N$ is the Nth remainder $\sum_N^\infty$ of the series (3.1), then

$$\int_0^\rho |\psi_N'(Re^{i\theta})|\, dR \leqq A\epsilon(\rho) \log^{1/2} (1/(1-\rho)),$$

with perhaps a different A. Integrating both sides of the inequality over E, we get

$$\int_0^\rho dR \int_E \left| \sum_{n=N+1}^{\infty} 2^n a_n R^{2^n} \exp 2^n i\theta \right| d\theta \leqq A\epsilon(\rho) \log^{1/2} (1/(1-\rho)). \tag{3.3}$$

Let us now assume without proof that for $N \geqq N_0(E)$, the inner integral on the left satisfies an inequality

$$\int_E \left| \sum_{n=N+1}^{\infty} 2^n a_n R^{2^n} \exp 2^n i\theta \right| d\theta \geqq B \left\{ \sum_{n=N+1}^{\infty} 2^{2n} a_n^2 R^{2^{n+1}} \right\}^{1/2}, \tag{3.4}$$

with $B = B(E)$, and let us fix such an N. It follows that

$$\int_0^\rho \left\{ \sum_{n=N+1}^{\infty} 2^{2n} a_n^2 R^{2^{n+1}} \right\}^{1/2} dR \leqq A\epsilon(\rho) \log^{1/2} (1/(1-\rho)).$$

Let $\rho_n = 1 - 2^{-n}$ $(n = 0, 1, 2, \cdots)$, $\rho = \rho_\nu$, $\nu > N+1$, and let us split the integral into a sum of integrals extended over the intervals (ρ_{n-1}, ρ_n) and a remaining interval. If in the integral over (ρ_{n-1}, ρ_n) we drop in the integrand all the terms except the nth, we get

$$\sum_{n=N+1}^{\nu} \int_{\rho_{n-1}}^{\rho_n} a_n 2^n R^{2^n} dR \leqq A\epsilon(\rho_\nu)\nu^{1/2},$$

so that

$$\sum_{n=N+1}^{\nu} a_n \leqq A\epsilon(\rho_\nu)\nu^{1/2}. \tag{3.5}$$

Let us now observe that, by Schwarz's inequality, the convergence of $\sum a_n^2$ implies $\sum_{N+1}^{\nu} a_n = o(\nu^{1/2})$, and that (as is very well known) this is the best possible estimate. In other words, given a sequence of positive numbers $\delta_n \to 0$, we may find the a's in such a way that $\sum a_n^2 < +\infty$ and that $\sum_{N+1}^{\nu} a_n \geqq \delta_\nu \nu^{1/2}$ for infinitely many ν's. Choosing the δ_ν in such a way that $\epsilon(\rho_\nu)/\delta_\nu \to 0$, we come to a contradiction with (3.5), so that for that particular sequence of the a's we cannot have (3.2) in a set of positive measure.

In order to close the gap in the above proof (cf. (3.4)) we need the following lemma.

LEMMA 2. *Let $\{b_n\}$ be any sequence of numbers such that $\sum |b_n|^2 = +\infty$ and that $\sum |b_n| \rho^{\lambda_n}$ converges for $0 \leqq \rho < 1$, where $\lambda_1 < \lambda_2 < \cdots$ are positive integers satisfying $\lambda_{n+1}/\lambda_n > q > 1$. Let E be any set of positive measure. Then there is an integer $N_0 = N(E, q)$, such that*

$$(3.6) \qquad \int_E \left| \sum_{n=N+1}^{\infty} b_n \rho^{\lambda_n} e^{i\lambda_n \theta} \right| d\theta \geqq B_{E,q} \left\{ \sum_{n=N+1}^{\infty} |b_n|^2 \rho^{2\lambda_n} \right\}^{1/2}$$

for $N > N_0(E, q)$ and all $0 \leqq \rho < 1$.

For it is well known that

$$(3.7) \qquad \int_E \left| \sum_{n=N+1}^{\infty} b_n \rho^{\lambda_n} e^{i\lambda_n \theta} \right|^2 d\theta \geqq \frac{|E|}{2} \sum_{n=N+1}^{\infty} |b_n|^2 \rho^{2\lambda_n},$$

provided that $N > N_0(E, q)$[28]. On the other hand, it is also well known[29] that

$$(3.8) \qquad \begin{aligned} \int_E \left| \sum_{n=N+1}^{\infty} b_n \rho^{\lambda_n} e^{i\lambda_n \theta} \right|^3 d\theta &\leqq \int_0^{2\pi} \left| \sum b_n \rho^{\lambda_n} e^{i\lambda_n \theta} \right|^3 d\theta \\ &\leqq A_q \left\{ \sum |b_n|^2 \rho^{2\lambda_n} \right\}^{3/2} \end{aligned}$$

for every N. Hence, applying to the integral of $|S|^2 = \left| \sum_{N+1}^{\infty} b_n \rho^{\lambda_n} e^{i\lambda_n \theta} \right|^2$ the inequality of Schwarz,

$$(3.9) \qquad \int_E |S|^2 d\theta = \int_E |S|^{1/2} |S|^{3/2} d\theta \leqq \left(\int_E |S| \, d\theta \right)^{1/2} \left(\int_E |S|^3 d\theta \right)^{1/2},$$

and using (3.7) and (3.8), we get (3.6). This completes the proof of Theorem 3.

4. Let us set

$$\left(\sum_{\nu=2^n+1}^{2^{n+1}} |c_\nu|^2 \right)^{1/2} = \gamma_n, \qquad n = 0, 1, 2, \cdots.$$

---

[28] See *TS.*, p. 122, where the corresponding result is established for general Toeplitz matrices.

[29] *TS*, p. 216.

The proof of Theorem 4 requires the following lemma.

LEMMA 3. *A necessary and sufficient condition for the convergence of the integral*

$$J = \int_0^1 \{\sum \nu^2 |c_\nu|^2 \rho^{2\nu}\}^{1/2} d\rho \tag{4.1}$$

is the finiteness of the sum

$$\Gamma = \gamma_0 + \gamma_1 + \gamma_2 + \cdots . \tag{4.2}$$

Let $\rho_n = 1 - 2^{-n}$, $I_n = (\rho_n, \rho_{n+1})$. Let $h(\rho)$ be the integrand in (4.1) and let J_n be the integral of $h(\rho)$ extended over I_n. Obviously

$$J = \sum_{n=0}^{\infty} J_n \geqq \sum_{n=0}^{\infty} \int_{\rho_n}^{\rho_{n+1}} \left(\sum_{\nu=2^n+1}^{2^{n+1}} \nu^2 |c_\nu|^2 \rho^{2\nu} \right)^{1/2} d\rho$$

$$\geqq \sum_{n=0}^{\infty} \rho_n^{2^{n+1}} 2^n \int_{\rho_n}^{\rho_{n+1}} \gamma_n d\rho \geqq A \sum_{n=0}^{\infty} \gamma_n.$$

This proves the necessity of the condition.

In the proof of sufficiency let us assume, as we may, that $c_0 = c_1 = 0$. Clearly,

$$J_k \leqq 2^{-k-1} h(\rho_{k+1}) \leqq 2^{-k-1} \sum_{n=0}^{\infty} \left\{ \sum_{\nu=2^n+1}^{2^{n+1}} \nu^2 |c_\nu|^2 \rho_{k+1}^{2\nu} \right\}^{1/2}$$

$$= 2^{-k-1} \sum_{n=0}^{k-1} + 2^{-k-1} \sum_{n=k}^{\infty} = P_k + Q_k,$$

say. Here

$$P_k \leqq 2^{-k-1} \sum_{n=0}^{k-1} 2^{n+1} \gamma_n = 2^{-k} \sum_{n=0}^{k-1} 2^n \gamma_n,$$

and, since $(1 - 1/m)^m \leqq e^{-1}$,

$$Q_k \leqq 2^{-k-1} \sum_{n=k}^{\infty} \gamma_n 2^{n+1} \exp(-2^{n-k-1}) = 2e \sum_{n=k}^{\infty} \gamma_n \xi_{n-k},$$

where $\xi_m = 2^m / \exp 2^{m-1}$. Observing that

$$\sum_{k=0}^{\infty} P_k \leqq \sum_{k=0}^{\infty} 2^{-k} \sum_{n=0}^{k-1} 2^n \gamma_n = \sum_{n=0}^{\infty} \gamma_n,$$

$$\sum_{k=0}^{\infty} Q_k \leqq 2 \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \gamma_n \xi_{n-k} = 2 \sum_{n=0}^{\infty} \gamma_n \sum_{k=0}^{n} \xi_{n-k} \leqq A \sum_{n=0}^{\infty} \gamma_n,$$

we deduce that $J = J_0 + J_1 + J_2 + \cdots \leqq A \sum_{n=0}^{\infty} \gamma_n$. This completes the proof of Lemma 3.

Let us now consider the integral

$$(4.3)\qquad \int_0^1\int_0^{2\pi} W(1,\theta;\phi_t)dtd\theta \leqq \int_0^1\int_0^{2\pi} dtd\theta\int_0^1 \left| \sum \nu c_\nu \psi_\nu(t)\rho^\nu e^{i\nu\theta}\right| d\rho.$$

If we integrate first with respect to t, and then with respect to θ, and also take into account that

$$\int_0^1 \left| \sum \alpha_\nu\psi_\nu(t)\right| dt \leqq A\left(\sum |\alpha_\nu|^2\right)^{1/2},$$

provided that $\sum|\alpha_\nu|^2<+\infty$, we obtain that

$$\int_0^1\int_0^{2\pi} W(1,\theta;\phi_t)d\theta dt \leqq A\int_0^1 \left(\sum \nu^2 |c_\nu|^2\rho^{2\nu}\right)^{1/2}d\rho \leqq A\sum \gamma_n.$$

Hence, if (4.2) is finite, $W(1, \theta; \phi_t)$ is finite almost everywhere in θ, t, and this proves the first part of Theorem 4.

For the proof of the second part we have to show that if $W(1, \theta; \phi_t)$ is finite for (θ, t) belonging to a (two-dimensional) set H of positive measure, the series $\gamma_0+\gamma_1+\cdots$ is finite. We may assume that $W(1,\theta;\phi_t)$ is bounded for $(\theta, t)\in H$. Hence,

$$\iint_H d\theta dt\int_0^1 \left|\sum_{\nu=0}^\infty \nu c_\nu\rho^\nu e^{i\nu\theta}\psi_\nu(t)\right| d\rho \leqq A.$$

By reducing H we may even assume that

$$(4.4)\qquad \int_0^1 d\rho\int_G d\theta\int_{E_\theta}\left|\sum_{\nu=0}^\infty \nu c_\nu\rho^\nu e^{i\nu\theta}\psi_\nu(t)\right| dt \leqq A,$$

where G is a set of positive measure, E_η denotes the intersection of H with the straight line $\theta=\eta$, and that the measure of each E_θ is positive.

We now observe that given a series $\sum\alpha_\nu\psi_\nu(t)$, with $\sum|\alpha_\nu|^2<+\infty$, and a set E of positive measure, we may find an integer $N_0=N_0(E)$, and a positive constant B_E such that

$$(4.5)\qquad \int_E\left|\sum_{\nu=N}^\infty \alpha_\nu\rho^\nu\psi_\nu(t)\right| dt \geqq B_E\left(\sum_{\nu=N}^\infty |\alpha_\nu|^2\rho^{2\nu}\right)^{1/2}, \qquad 0\leqq\rho<1,$$

for $N>N_0$. The proof is exactly the same as that of Lemma 2. What is more, if we replace α_ν by $\alpha_\nu e^{i\nu\theta}$, (4.5) will still be valid with the same N_0 and B_E. Let us now consider (4.4), where we integrate first with respect to t, over the set E_θ. For every $\theta\in G$ we may find the corresponding $N_0(E_\theta)=N_0(\theta)$ and $B_{E_\theta}=B(\theta)$. Hence we may find a subset G' of G which is of positive measure and over which $N_0(\theta)$ is bounded above, say not greater than N', and $B(\theta)$ is bounded below, say not less than $B'>0$. From (4.4) we obtain at once that

$$|G'|\cdot B'\int_0^1\left(\sum_{\nu=N'}^{\infty}\nu^2|c_\nu|^2\rho^{2\nu}\right)^{1/2}d\rho \leqq A.$$

On account of Lemma 3, this shows that the series $\gamma_0+\gamma_1+\cdots$ is convergent. This concludes the proof of Theorem 4.

5. Theorem 4 remains valid if, instead of the analytic functions $\psi_t(z) = \sum c_\nu z^\nu\psi_\nu(t)$, we consider the family of harmonic functions

$$h_t(\rho,\theta) = \sum_{\nu=1}^{\infty}(a_\nu\cos\nu\theta + b_\nu\sin\nu\theta)\rho^\nu\psi_\nu(t).$$

The finiteness of the series (1.6) will now be replaced by the finiteness of the sum

$$\sum_{n=0}^{\infty}\left\{\sum_{\nu=2^n+1}^{2^{n+1}}(a_\nu^2+b_\nu^2)\right\}^{1/2}. \tag{5.1}$$

Let $W_t(1,\theta)$ denote the variation of the function $h_t(\rho,\theta)$ over the radius $(0, e^{i\theta})$. The fact that $W_t(1,\theta)$ is finite almost everywhere in θ and t, if (5.1) is finite, is contained in the first part of Theorem 4. The fact that $W_t(1,\theta)$ is almost always infinite in the θ, t-plane, if the series (5.1) is divergent, may be proved in exactly the same way as the second part of Theorem 4, provided we use the following lemma.

LEMMA 4. *Let $A_n(\theta)=a_n\cos n\theta+b_n\sin n\theta$, $n=1, 2, \cdots$. If the series*

$$\sum_{n=0}^{\infty}\left\{\sum_{\nu=2^n+1}^{2^{n+1}}A_\nu^2(\theta)\right\}^{1/2} \tag{5.2}$$

converges for every θ belonging to a set E of positive measure, then the series (5.1) *converges.*

Let us set

$$\alpha_n = \left\{\sum_{\nu=2^n+1}^{2^{n+1}}(a_\nu^2+b_\nu^2)\right\}^{1/2}, \qquad \beta_n(\theta) = \alpha_n^{-1}\left\{\sum_{\nu=2^n+1}^{2^{n+1}}A_\nu^2(\theta)\right\}^{1/2},$$

so that the series (5.2) (with $a_0=a_1=b_1=0$) may be written

$$\sum_{n=0}^{\infty}\alpha_n\beta_n(\theta). \tag{5.3}$$

It is enough to show that

$$\int_E\beta_n(\theta)d\theta \geqq \epsilon = \epsilon(E) > 0 \qquad (n = 0, 1, 2, \cdots) \tag{5.4}$$

for every set E of positive measure. For without loss of generality of Lemma 4 we may assume that the series (5.2) is bounded, say not greater than M, over E. Integrating (5.3) over E, we get

$$\sum_{n=0}^{\infty} \alpha_n \int_E \beta_n d\theta \leqq M \mid E \mid,$$

and the convergence of (5.1) follows at once.

In order to prove (5.4) we note that

$$\int_E \beta_n^2 d\theta \to \mid E \mid /2. \tag{5.5}$$

This follows at once if we multiply out $A_\nu^2(\theta)$, replace $\cos^2 \nu\theta$ and $\sin^2 \nu\theta$ by $(1 \pm \cos 2\nu\theta)/2$, and use the fact that the Fourier coefficients of the characteristic function of the set E tend to 0. Since

$$\int_E \beta_n^2 d\theta \leqq \max_\theta \beta_n \cdot \int_E \beta_n d\theta,$$

we see that $\liminf \int_E \beta_n d\theta \geqq |E|/2$, which proves (5.4).

It is clear that Lemma 5 might be generalized in various ways, but this is of no interest to us.

References

1. A. E. Ingham, *On the "high-indices" theorem of Hardy and Littlewood*, Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 1–8.

2. G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math. vol. 48 (1930) pp. 81–116.

3. ———, *The strong summability of Fourier series*, Fund. Math. vol. 25 (1935) pp. 162–189.

4. ———, *Some new properties of Fourier constants*, Math. Ann. vol. 97 (1926) pp. 159–209.

5. J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series*. II, Proc. London Math. Soc. vol. 42 (1937) pp. 52–89.

6. J. Marcinkiewicz, *Sur quelques intégrales du type de Dini*, Annales de la Société Polonaise de Mathématiques vol. 17 (1938) pp. 42–50.

7. J. Marcinkiewicz and A. Zygmund, *On a theorem of Lusin*, Duke Math. J. vol. 4 (1938) pp. 473–485.

8. M. Riesz, *Sur les séries conjuguées*, Math. Zeit. vol. 27 (1927) pp. 218–244.

9. D. C. Spencer, *A function-theoretic identity*, Amer. J. Math. vol. 65 (1943) pp. 147–160.

10. A. Zygmund, *On the convergence and summability of power series on the circle of convergence*. I, Fund. Math. vol. 30 (1938) pp. 170–196.

11. ———, *A theorem on generalized derivatives*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 917–923.

12. ———, *Trigonometrical series*, Warsaw, 1935.

Mt. Holyoke College,
South Hadley, Mass.